Mon théorème préféré : Le lemme de Breiman

– Histoire et applications modernes –

Thomas Opitz

BioSP–INRAE

Au sommet de Rochebrune

25 mars 2026

Léo Breiman

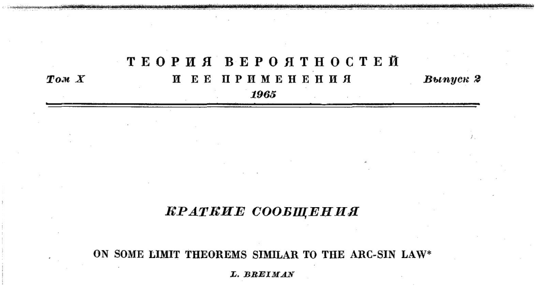
- Born January 27, 1928, New York City
- Degree in Physics 1949
- *He then decided to study philosophy at Columbia University, but was persuaded to return to mathematics when the head of the philosophy department confided that his two best Ph.D.s couldn't find jobs. He received a master's degree in mathematics from Columbia in 1950.* (Obituary)
- University of California (Los Angeles / Berkeley)
- Retired 1993
- Died July 5, 2005, Berkeley

What is he known for?

Groundbreaking contributions in statistical learning

- Classification and Regression Trees (1984, ∼70,000 citations GS)
- Random Forests (2001, ∼180,000 citations GS)
- Many other highly influential works for machine learning
- **But also this early paper:**
  Breiman, L. (1965). On some limit theorems similar to the arc-sin law. *Theory of Probability & Its Applications*, 10(2), 323-331.
  (∼600 citations GS)

  containing an auxiliary result known as **Breiman's lemma**.



ТЕОРИЯ ВЕРОЯТНОСТЕЙ
И ЕЕ ПРИМЕНЕНИЯ

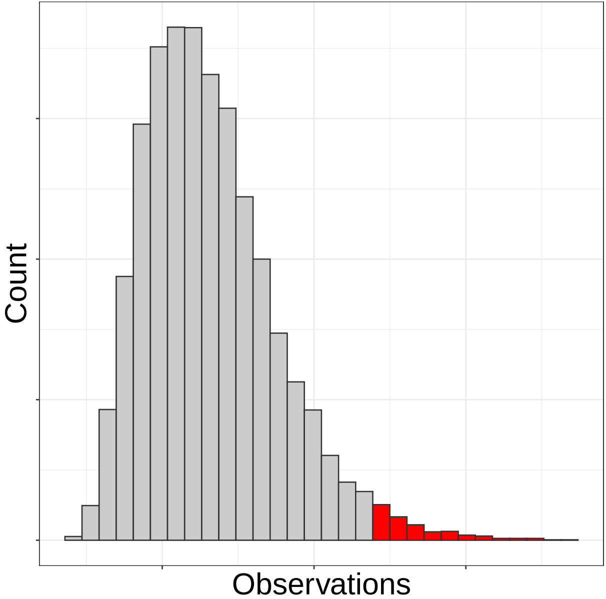
Том X — *1965* — *Выпуск 2*

КРАТКИЕ СООБЩЕНИЯ

ON SOME LIMIT THEOREMS SIMILAR TO THE ARC-SIN LAW*

L. BREIMAN

Heavy tails


Count
Observations

Tail probabilities of a product of a heavy-tailed and a lighter-tailed variable?

Theorem (Breiman's lemma)

Let X and Y be independent nonnegative random variables. Assume that X has a regularly varying tail with index $\alpha > 0$, i.e.,

$$\mathbb{P}(X > x) = x^{-\alpha} L(x), \quad x \to \infty,$$

where L is a slowly varying function, i.e., $L(tx)/L(t) \to 1$, $t \to \infty$.
Suppose further that

$$\mathbb{E}[Y^{\alpha+\varepsilon}] < \infty \quad \text{for some } \varepsilon > 0.$$

Then,

$$\mathbb{P}(XY > x) \sim \mathbb{E}[Y^{\alpha}] \times \mathbb{P}(X > x), \quad x \to \infty.$$

Regularly varying = Power law
(t, F (Fisher), Pareto, Fréchet...)

Equivalently, $\overline{F}(tx)/\overline{F}(t) \to x^{-\alpha}$ if $X \sim F$, with the survival function $\overline{F}(x) = \mathbb{P}(X > x)$

Tail probabilities of the sum of an exponential-tailed and a lighter-tailed variable?

- Consider $X \sim F$ exponential-tailed with rate $\alpha > 0$, i.e.,

$$\overline{F}(x+t)/\overline{F}(t) \to \exp(-\alpha x), \quad t \to \infty$$

(Exponential, Gamma, Inverse Gaussian...)

- Assume moment condition on Y: $\mathbb{E}[\exp((\alpha+\epsilon)Y)] < \infty$
- **Result:**

$$\mathbb{P}(X+Y>x)\times \sim \mathbb{E}[\exp(\alpha Y)] \times \mathbb{P}(X>x), \quad x \to \infty$$

⚠ Behavior of $X+Y$ would be quite different for lighter tails (e.g. X, Y Gaussian)!

Sketch of proof (multiplicative version)

- **Rewrite as expectation**: $\mathbb{P}(XY > x) = \mathbb{E}\big[\mathbb{P}(X > x/Y \mid Y)\big] = \mathbb{E}\big[\bar{F}_X(x/Y)\big]$
- **Use regular variation:** $\bar{F}_X(x) = x^{-\alpha}L(x) \Rightarrow \bar{F}_X(x/Y) = x^{-\alpha}Y^{\alpha}L(x/Y)$

- **Factorize for tail probability of** X**:**

$$\mathbb{P}(XY > x) = x^{-\alpha}L(x)\mathbb{E}\left[Y^{\alpha}\frac{L(x/Y)}{L(x)}\right] = \bar{F}(x) \times \mathbb{E}\left[Y^{\alpha}\frac{L(x/Y)}{L(x)}\right]$$

- **Slow variation** of $L \Rightarrow$ **Potter bounds** $\frac{L(x/Y)}{L(x)} \leq C_{\varepsilon}(Y^{\epsilon} + Y^{-\varepsilon})$, for any $\varepsilon > 0$

  $\Rightarrow$ **Obtain integrable bound** $Y^{\alpha}\frac{L(x/Y)}{L(x)} \leq C_{\varepsilon}(Y^{\alpha+\varepsilon} + Y^{\alpha-\varepsilon})$

- **Exploit moment condition** $\mathbb{E}[Y^{\alpha+\varepsilon}] < \infty$ **for dominated convergence**:

$$\mathbb{E}\left[Y^{\alpha}\frac{L(x/Y)}{L(x)}\right] \to \mathbb{E}[Y^{\alpha}], \quad x \to \infty$$

- **We get the desired result:** $\mathbb{P}(XY > x) \sim \mathbb{E}[Y^{\alpha}] \times \mathbb{P}(X > x)$

Application example 1: Asymptotics for joint tail probabilities

Common factor model. Consider a random vector with components

$$Z_j = X \times Y_j, \quad j = 1, \ldots, d,$$

with X, Y_j mutually independent, fulfilling the conditions of Breiman's lemma.

Then,

$$\begin{aligned}\Pr(Z_j > tu_j \text{ for all } j) &= \Pr(\min_j (Z_j/u_j) > t) \\ &= \Pr(X \times \min_j (Y_j/u_j) > t) \\ &\sim \Pr(X > t) \times \mathbb{E}[\min_j (Y_j/u_j)^\alpha]\end{aligned}$$

i.e.,

$$\frac{\Pr(Z_j > tu_j \text{ for all } j)}{\Pr(X > t)} \to \mathbb{E}[\min_j (Y_j/u_j)^\alpha]$$

Similarly,

$$\frac{\Pr(Z_j > tu_j \text{ for at least one } j)}{\Pr(X > t)} \to \mathbb{E}[\max_j (Y_j/u_j)^\alpha]$$

Definition of tail correlation

For $X_1 \sim F_1, X_2 \sim F_2$, consider

$$\chi(u) = \Pr\left(F_2(X_2) > u \mid F_1(X_1) > u\right) = \frac{\Pr\left(F_2(X_2) > u, F_1(X_1) > u\right)}{\Pr\left(F_1(X_1) > u\right)}, \quad u \in (0,1)$$

Tail correlation (or **χ-coefficient**): $\chi = \lim_{u \to 1} \chi(u) \in [0,1]$

Characterizing concomitance of extremes:

- $\chi > 0 \Rightarrow$ **Asymptotic dependence** (**Simultaneous extremes**)
- $\chi = 0 \Rightarrow$ **Asymptotic independence** (**Independent extremes**)

What is the tail correlation in common factor model $(Z_1, Z_2) = X \times (Y_1, Y_2)$?

$$\Rightarrow \chi = \mathbb{E}\left[\min_j \frac{Y_j^\alpha}{\mathbb{E}[Y_j^\alpha)]}\right]$$

Application example 3: Models with transition in tail dependence regime

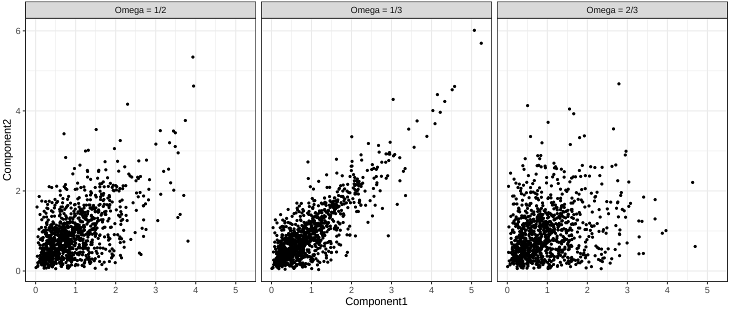
Consider i.i.d. exponential-tailed X_j with $\alpha = 1$, $j = 0, 1, 2$
Common factor model with weight $\omega \in [0, 1]$:

$$Z_1 = (1-\omega)X_0 + \omega \times X_1$$
$$Z_2 = (1-\omega)X_0 + \omega \times X_2$$

$\Rightarrow$ Interpolates between perfect dependence ($\omega = 0$) and independence ($\omega = 1$)

- **If** $\omega < 1/2 \Rightarrow$ Tail of X_0 dominates $\Rightarrow \chi > 0 \Rightarrow$ **Simultaneous extremes**
- **If** $\omega > 1/2 \Rightarrow \chi = 0 \Rightarrow$ Tail of X_1 and X_2 dominates $\Rightarrow$ **Independent extremes**
- **Special case** $X_j \sim$ **Exp**$(1) \Rightarrow \chi = \frac{1-2\omega}{1-3\omega/2}$, $\chi = 0$ if $\omega = 1/2$, $\chi = 1$ if $\omega = 0$

$\Rightarrow$ Transition from simultaneous to independent extremes at $\omega = 1/2$

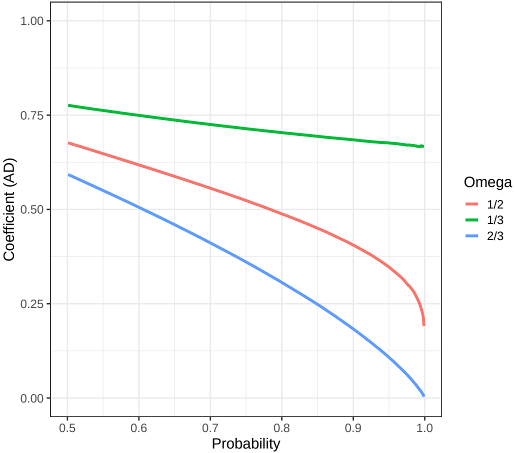


Recall:

$$Z_1 = (1-\omega)X_0 + \omega \times X_1$$
$$Z_2 = (1-\omega)X_0 + \omega \times X_2$$

where $X_0, X_1, X_2 \sim \mathrm{Exp}(1)$ and $\omega \in \{1/3, 1/2, 2/3\}$

Tail correlations $\chi(u)$



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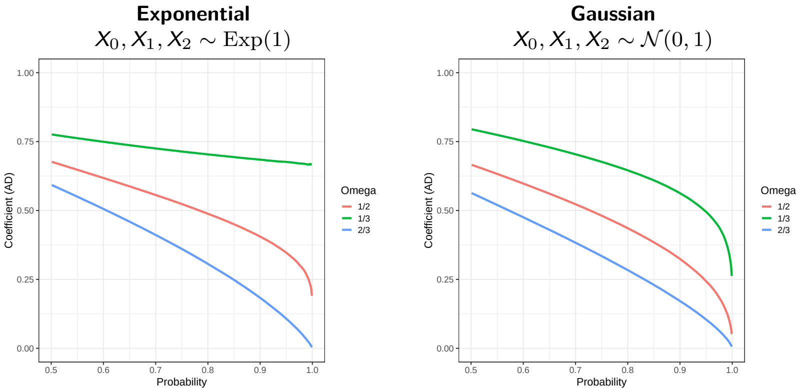

⚠ **Nondegenerate Gaussian models never have simultaneous extremes!**

Application example 4: Tail correlation in general linear exponential systems

Consider $X_j \overset{\text{ind.}}{\sim} F_j \in \mathrm{ET}_{\beta_j}$ (where $\beta_j = \infty$ if tail is lighter than exponential).

$$Z_1 = \omega_{11} X_1 + \ldots + \omega_{1J} X_J$$
$$Z_2 = \omega_{21} X_1 + \ldots + \omega_{2J} X_J,$$

- $\tilde{\omega}_{ij} = \omega_{ij,+}/\beta_j =$ scale parameter (i.e., inverse rate) of $\omega_{ij} X_j$.
- Maximum scales $\omega_i^\star = \max_j \tilde{\omega}_{ij}$, $i = 1, 2$
- Indices where the maximum is realized: $I_i = \{j : \tilde{\omega}_{ij} = \omega_i^\star\}$.

Results

1. Suppose that $I_1 = I_2$ and $\omega_i^\star > 0$, $i = 1, 2$. Then, the variables Z_1 and Z_2 are asymptotically dependent with

$$\chi = \mathbb{E}\left[\min\left(\exp(\tilde{X}_1)/m_1, \exp(\tilde{X}_2)/m_2\right)\right] > 0,$$

   where $\tilde{X}_i = \sum_{j \notin I_i} X_j \omega_{ij}/\omega_i^\star$ and $m_i = \mathbb{E}[\exp(\tilde{X}_i)]$, $i = 1, 2$.

2. If $I_1 \cap I_2 = \emptyset$, then $\chi_{12} = 0$.

Impact

- Heavy-tail theory
- Classical Multivariate Extreme-Value Theory
- More flexible extremal dependence than Gaussian copula
(*The formula that killed Wall Street* – Financial crisis 2008)
- Many important results in Quantitative Risk Management

Towards generative AI models with tractable and flexible extremal dependence?