Processus de Hawkes avec covariables pour modéliser des comportements de narvals (monodon monoceros)

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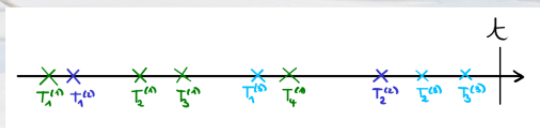
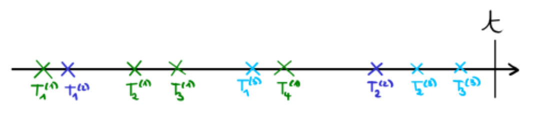
Background image from WWF *https://wwf.ca/fr/species/narval/*

Point process and intensity



- Point process: $\mathrm{N} = (\mathrm{N}^j)_{1 \leqslant j \leqslant d}$
- Observations: Ordered event times $(T_k^j)_{1 \leqslant j \leqslant d, k \geqslant 1}$
- Intensity function: for $1 \leqslant j \leqslant d$ let us define $\lambda^{(j)}(t)$ the instantaneous probability of observing an event of type j at time t

Multidimensional Hawkes process



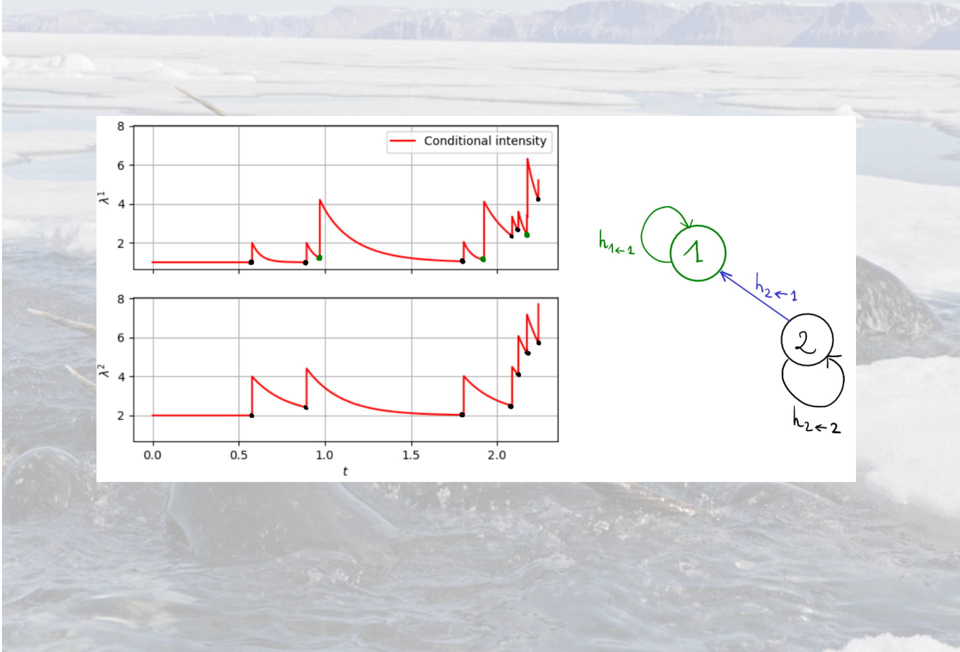
Intensity

For $1 \leqslant i \leqslant d$,

$$\lambda^{(i)}(t) = \mu_i + \sum_{j=1}^{d} \sum_{T_k^j \leqslant t} h_{i \leftarrow j}(t - T_k^j)$$

- μ_i is a baseline coefficient (one for each dimension of the process)
- $h_{i \leftarrow j}$ describes the influence of events of type j on events of type i ($h_{i \leftarrow j} \geqslant 0$)

Example of bidimensional Hawkes process



Accounting for external covariates

Observations:

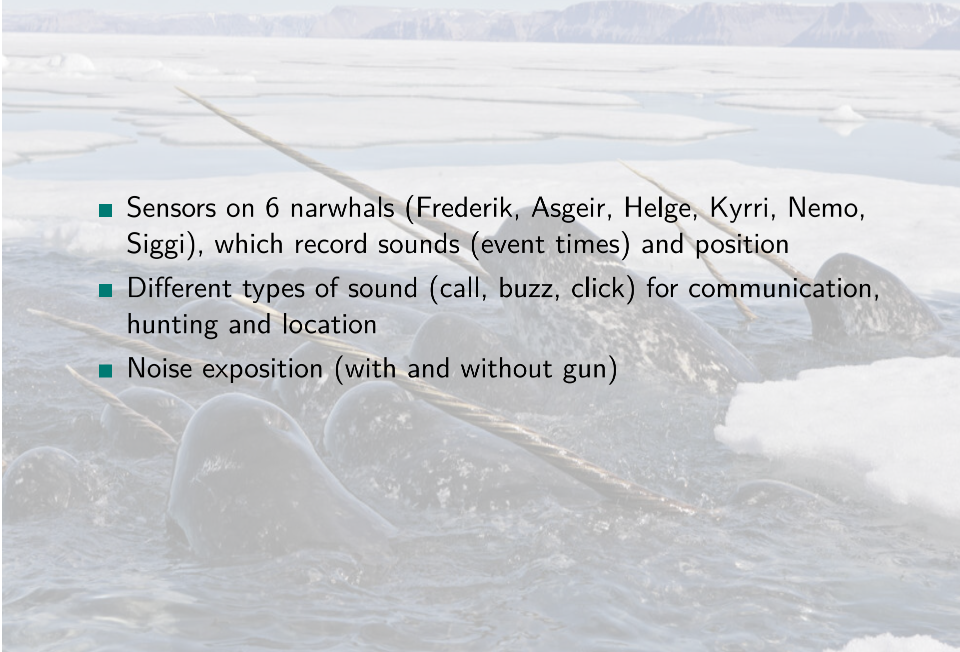
- Event times $(T_k^j)_{1 \leqslant j \leqslant d, k \geqslant 1}$
- X_t external covariate observed at (almost) continuous time t (typically spatial information)

Intensity

For $1 \leqslant i \leqslant d$,

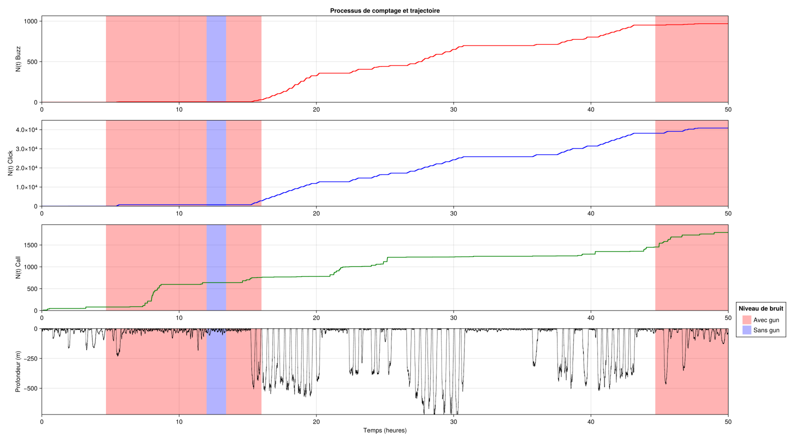
$$\lambda^{(i)}(t) = g_{\mu_i}(X_t) + \sum_{j=1}^{d} \sum_{T_k^j \leqslant t} h_{i \leftarrow j}(t - T_k^j, X_t)$$

Narwhals monitoring in Greenland



- Sensors on 6 narwhals (Frederik, Asgeir, Helge, Kyrri, Nemo, Siggi), which record sounds (event times) and position
- Different types of sound (call, buzz, click) for communication, hunting and location
- Noise exposition (with and without gun)

Example of data (Frederik)



- Measures of each type of sounds (call, buzz, click)
- Measure of the trajectory (depth)
- Different phases of noise exposition

Questions

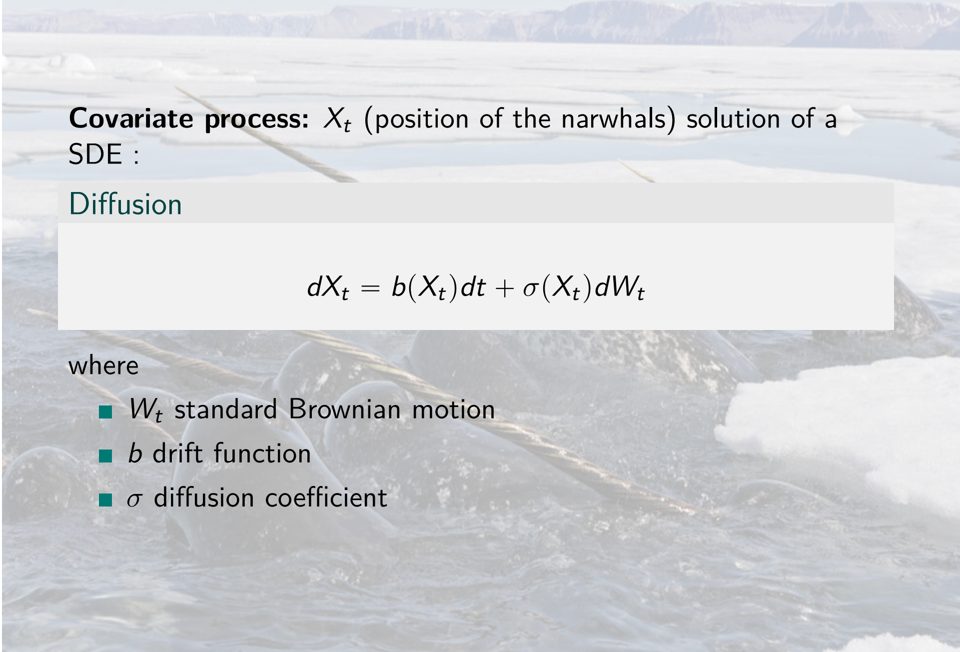
Methodological questions:

- Statistical inference
- Interest of accounting for the covariates
- Fit between the model and the data

Application:

- Identify patterns between different types of calls
- Study the impact of external effects
- Distinguish between the effect of the noise on the trajectory and on the calls

Covariate process



Covariate process: X_t (position of the narwhals) solution of a SDE :

Diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

where

- W_t standard Brownian motion
- b drift function
- σ diffusion coefficient

(Conditional) Intensity

For $1 \leqslant i \leqslant 3$,

$$\lambda^{(i)}(t) = g_{\mu_i}(X_t) + \sum_{j=1}^{d} \sum_{T_k^j \leqslant t} h_{i \leftarrow j}(t - T_k^j)$$

In our case:

- Each dimension corresponds to a type of sound ($d = 3$)
- X_t described the position (depth) of the narwhal at time t
- X_t impacts the baseline but not the memory of the process
- X_t impacts the intensity of the Hawkes process but not the other way around
- We focus on the estimation of the Hawkes process and not on the diffusion process

Maximum likelihood

- Exponential kernels $h_{i \leftarrow j}(t) = \alpha_{ij} \exp(-\beta_{ij} t)$
- Parameter $\theta = ((\mu_i)_{1 \leqslant i \leqslant d}, (\alpha_{ij})_{1 \leqslant i,j \leqslant d}, (\beta_{ij})_{1 \leqslant i,j \leqslant d})$

Log-likelihood of a point process

$$\ell_T(\theta) = \sum_{i=1}^{d} \left(\int_0^T \log \lambda_{i,\theta}(t) \mathrm{d}\mathrm{N}_t^i - \int_0^T \lambda_{i,\theta}(t) \, \mathrm{d}t \right)$$

- The covariate process appears in $\lambda_{i,\theta}(t)$ since
  $\lambda^{(i)}(t) = g_{\mu_i}(X_t) + \sum_{i=1}^{M} \sum_{T_k^j \leqslant t} \alpha_{ij} exp(\beta_{ij}(t - T_k^j))$
- We do not estimate the parameters of the diffusion process

Properties of the MLE

Asymptotic normality

Under identifiability and ergodicity conditions,

$$\sqrt{T}(\widehat{\theta}_T - \theta^\star) \xrightarrow[T \to \infty]{\mathcal{L}} \mathcal{N}(0, \Gamma^{-1}),$$

where $\widehat{\theta}_T$ is the MLE and Γ the Fisher information matrix.

- The proof relies on the general result for the convergence of the MLE for point processes (Clinet and Yoshida, 2017)
- Conditions satisfied for the Hawkes process with constant baseline
- Additional assumptions when the baseline is stochastic: boundedness, regularity and identifiability of the baseline function + ergodicity of the covariate process

Test procedure

Test for one coefficient $\mathbf{H}_0 : \theta_i^\star = \theta_{0,i}$ vs $\mathbf{H}_1 : \theta_i^\star \neq \theta_{0,i}$

Test statistic:

$$Z_i = \frac{\sqrt{T}(\widehat{\theta}_{T,i} - \theta_{0,i})}{\widehat{\sigma}_i} \underset{H_0}{\simeq} \mathcal{N}(0,1), \quad \text{with } \widehat{\sigma}_i = \sqrt{(\widehat{\Gamma}^{-1})_{ii}},$$

- Test on the interaction coefficients $\alpha_{ij} = 0$ vs $\alpha_{ij} \neq 0$
- Test on the impact of the covariate $\mu_i = 0$ vs $\mu_i \neq 0$ (with an appropriate parameterization of g_{μ_i})

Numerical example

Diffusion process Ornstein–Uhlenbeck

$$dX_t = -\xi X_t dt + \sqrt{|2\xi|} dW_t,$$

with $\xi = 0.05$ and $X_0 = (0,0)$.

Hawkes exponential kernel with $\alpha^\star = \begin{pmatrix} 0.3 & 0.4 \\ 0.5 & 0.4 \end{pmatrix}$ and $\beta^\star = (0.8, 1.5)$.

True baseline functions:

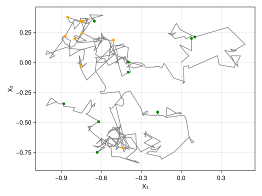
- $g_{\mu_1^\star}(x) = 0.8 + (0.5 - 0.8)\exp(-5\|x - (0.1, 0.1)\|^2)$ (space-dependent)
- $g_{\mu_2^\star}(x) = 0.7$ (constant)

Baseline functions used for estimation:

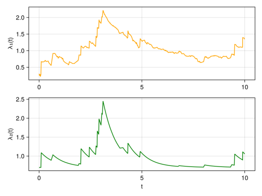
- $g_{\mu_1^\star}(x) = \mu_{12} + (\mu_{11} - \mu_{12})\exp(-5\|x - (0.1, 0.1))\|^2)$ (space-dependent)
- $g_{\mu_2^\star}(x) = \mu_{21} + (\mu_{21} - \mu_{22})\exp(-5\|x - (0.1, 0.1))\|^2)$ (space-dependent)

Numerical results

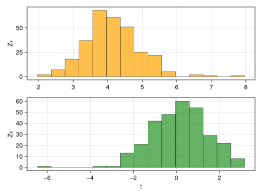
Test: $H_0 : \mu_{i1} = \mu_{i2}$ (constant baseline) vs $H_1 : \mu_{i1} \neq \mu_{i2}$ (space-dependent baseline)



(a) Example of (N, X)



(b) Intensity process



(c) Test statistic distribution

- The test accurately rejects H_0 when the process depends on X (with 300 repetitions and $T = 2000$)

Back to narwhals

Main practical challenges

- Include inhibition
- Choose an appropriate baseline function (ongoing)
  $\hookrightarrow$ Nonparametric decomposition of the baseline function

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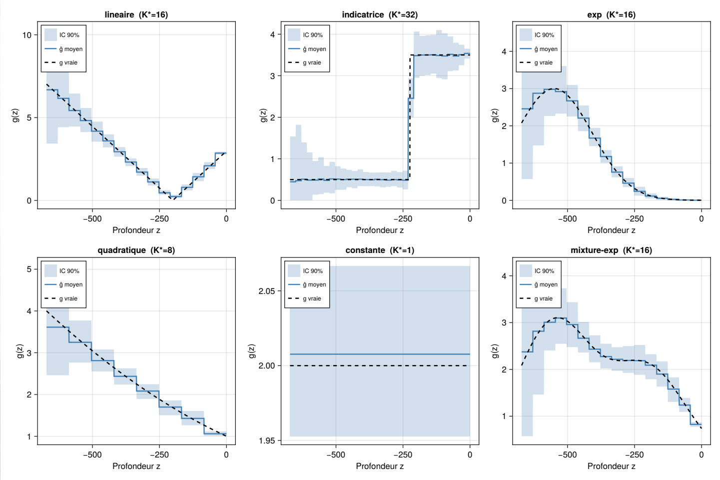
Preliminary study:

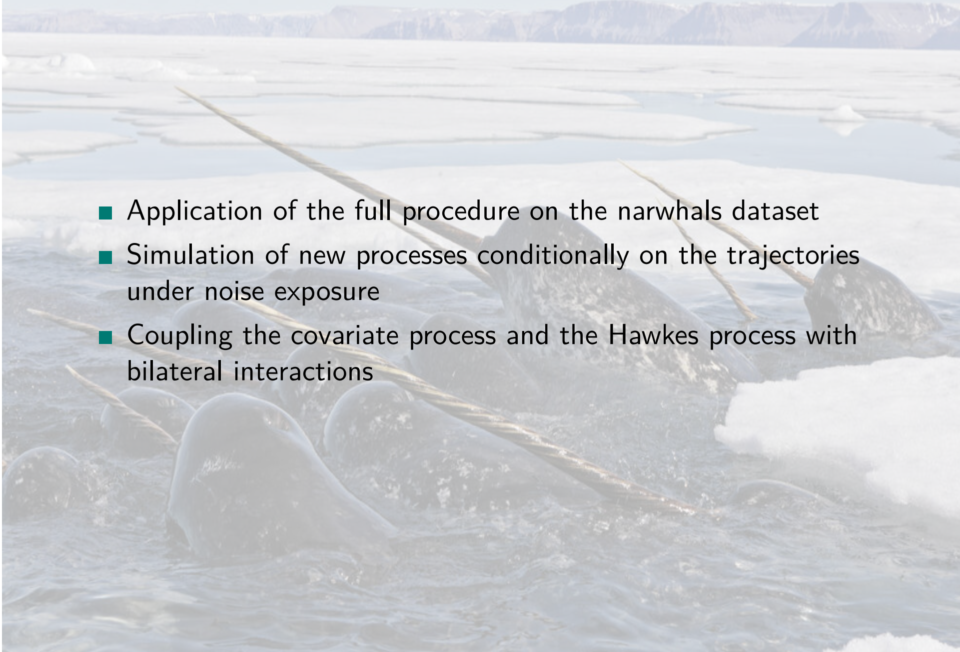
- Consider a Poisson process $\lambda(t) = g(X_t)$, where X_t are the real narwhals' trajectories and g different types of functions (linear, quadratic, exponential, Gaussian mixture...)
- Nonparametric estimation of g on a histogram basis
- Empirical calibration for the size of the basis

Preliminary results



- Accurate estimation with larger variability for deepest points of the trajectory
- Investigation of automatic criteria for the basis calibration

Perspectives



- Application of the full procedure on the narwhals dataset
- Simulation of new processes conditionally on the trajectories under noise exposure
- Coupling the covariate process and the Hawkes process with bilateral interactions