# Joint stochastic simulation of extreme coastal and offshore significant wave heights

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27/03/2024

Statistiques au sommet de Rochebrune

Université de Bretagne Occidentale

LMBA

# Motivation

Mediterranean storm - Oct 2018, ©Ville de Menton

170 km/h

up to 7 metres

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Numerous **impacts** of extreme maritime events: coastal flood hazard, coastal erosion, reliability of offshore and coastal structures

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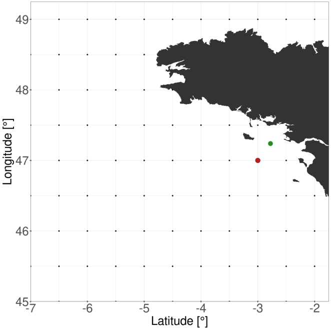
**How to generate extreme coastal wave events given covariates?**

⇝ Focus on a **stochastic** simulator that, given some offshore sea state conditions, can generate extreme wave height near the coast

# Data considered



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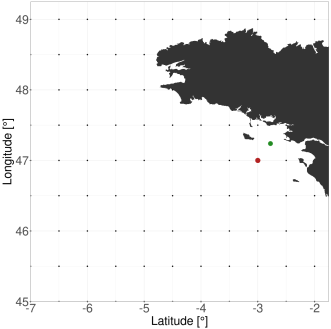


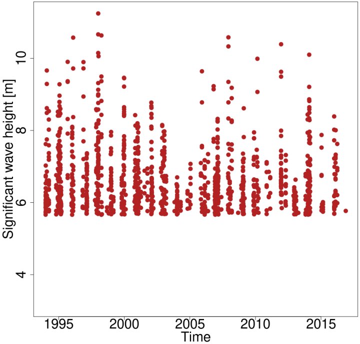


- Offshore significant wave height $H_o$

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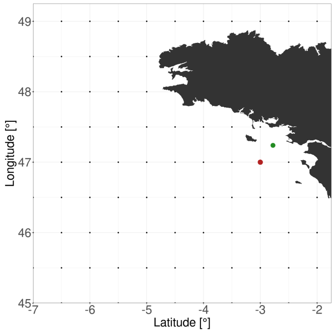


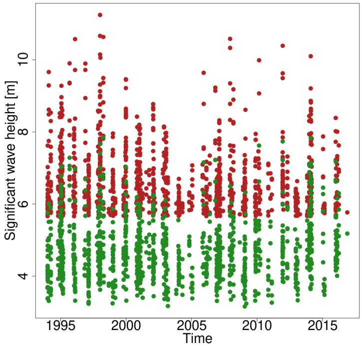


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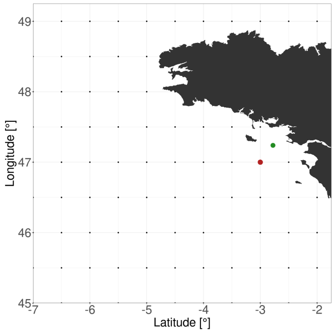


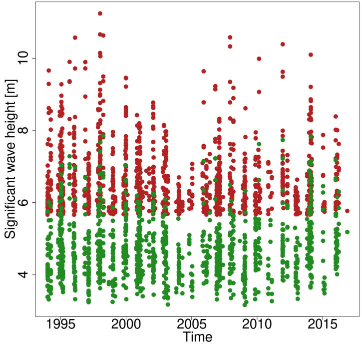


- Offshore significant wave height $H_o$
- Coastal significant wave height $H_c$
- Time resolution: **3-hour** time scale, from **1994** to **2016**

## Wave energy dissipation and simulation framework

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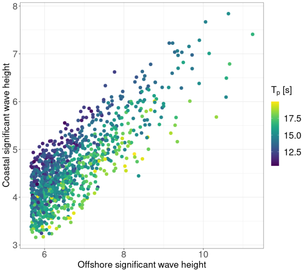

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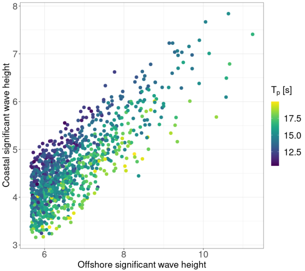


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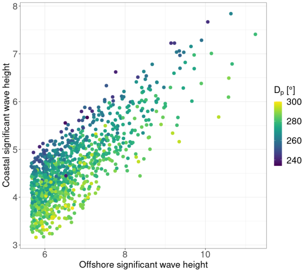

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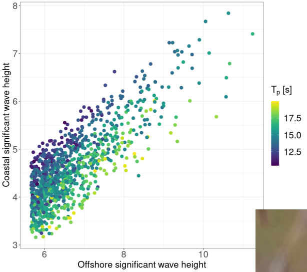


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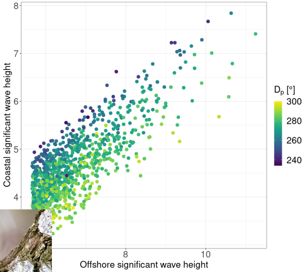


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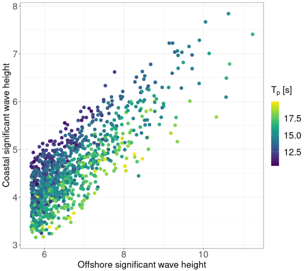


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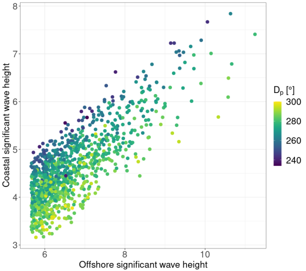


| | $H_c$ | $H_o$ |
|---|---|---|
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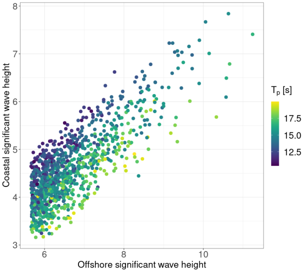


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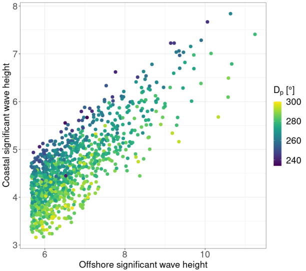


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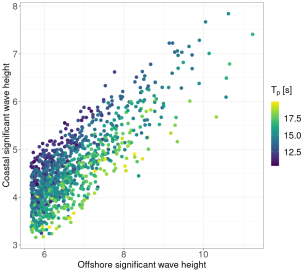


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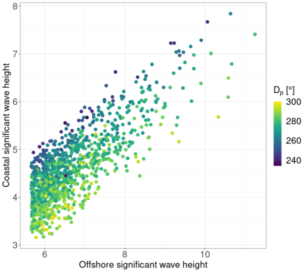


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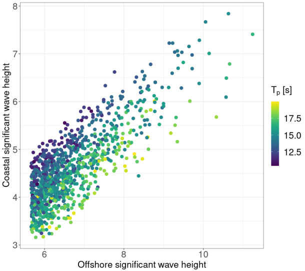


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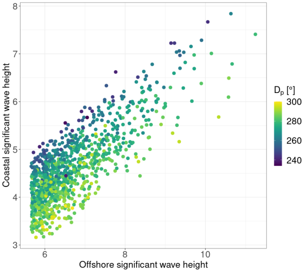


| | $H_c$ | $H_o$ | $D_p$ | $T_p$ |
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| **Joint simulation** | x | x | v | v |
| **Conditional simulation** | x | v | v | v |

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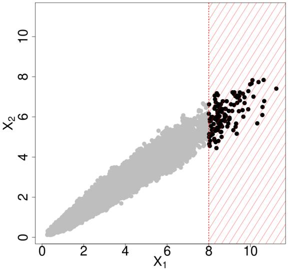
In the metocean community, mainly based on the conditional model of Heffernan and Tawn (2004):

$$X_2 \mid X_1 = x, \text{ for } x > u_1$$



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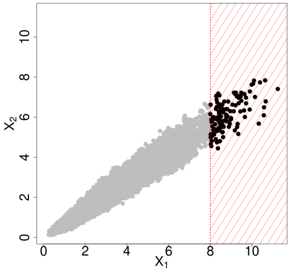
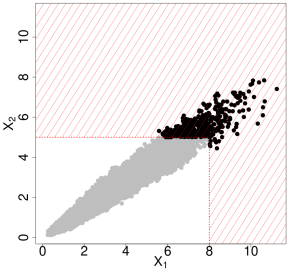
Our approach is rather based on **bivariate peaks over threshold** modelling Rootzén and Tajvidi (2006)

$$(X_1 - u_1, X_2 - u_2) \mid X_1 > u_1 \text{ or } X_2 > u_2$$



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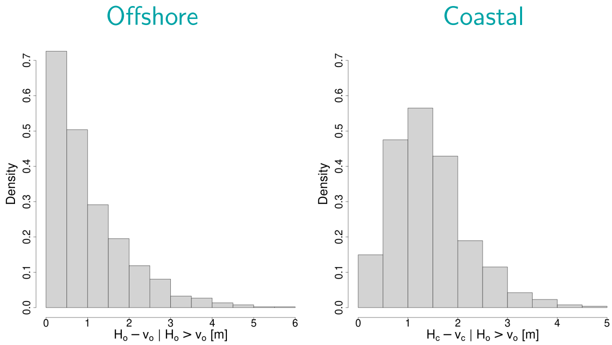
Key steps:

1. Marginal conditional modelling



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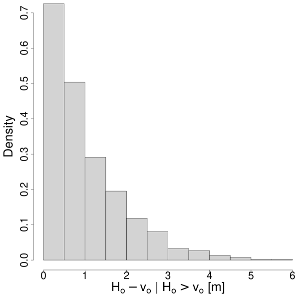
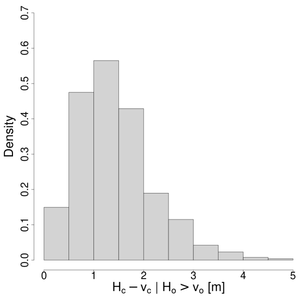
Key steps:

1. Marginal conditional modelling within the class of **extended generalised Pareto distributions** Naveau et al. (2016) and Le Carrer (2022)

Offshore




Coastal




$$\begin{cases} \mathbb{P}(H_o - v_o \leq x \mid H_o > v_o, T_p, D_p) = \left(1 - \left(1 + \dfrac{\xi_o x}{\sigma_o(T_p, D_p)}\right)^{-1/\xi_o}\right)^{\kappa_o} \\ \mathbb{P}(H_c - v_c \leq x \mid H_o > v_o, T_p, D_p) = \left(1 - \left(1 + \dfrac{\xi_c x}{\sigma_c(T_p, D_p)}\right)^{-1/\xi_c}\right)^{\kappa_c} \end{cases} \quad (1)$$

## Methodology to model $(H_o, H_c) \mid H_o > v_o, T_p, D_p$

Key steps:

1. Marginal conditional modelling within the class of **extended generalised Pareto distributions** Naveau et al. (2016) and Le Carrer (2022)
2. Transformation to common exponential margins

$$\begin{aligned} H_o^E &:= -\log\left\{1 - \hat{F}_o[(H_o - v_o)/\hat{\sigma}_o(T_p, D_p)]\right\} \\ H_c^E &:= -\log\left\{1 - \hat{F}_c[(H_c - v_c)/\hat{\sigma}_c(T_p, D_p)]\right\} \end{aligned} \quad (2)$$

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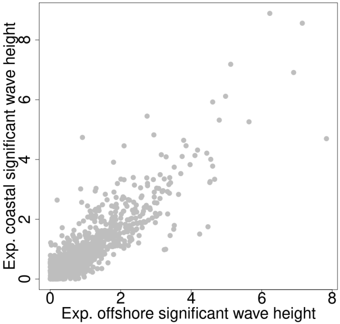
4. **Non-parametric simulation of bivariate extreme** $H_s$ within the class of MGP distributions

# Bivariate Pareto model



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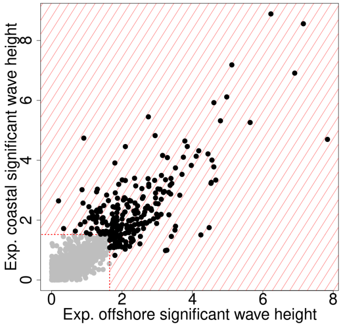


$$(Z_1, Z_2) := \left[\left(H_o^E - u_o, H_c^E - u_c\right) \mid H_o^E > u_o \text{ or } H_c^E > u_c\right] \tag{3}$$

where $(u_o, u_c) \in \mathbb{R}_+^2$

## Stochastic generator

- Stochastic representation of Rootzén et al. (2018):

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = E + \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} - \max(T_1, T_2)$$

with $E \sim Exp(1)$ and $T_1, T_2$ independent of $E$

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- **Simple rewriting** with $\Delta := Z_1 - Z_2$

$$\begin{cases} Z_1 = E + \Delta \mathbb{1}_{\Delta < 0}, \\ Z_2 = E - \Delta \mathbb{1}_{\Delta \geq 0} \end{cases} \qquad (4)$$

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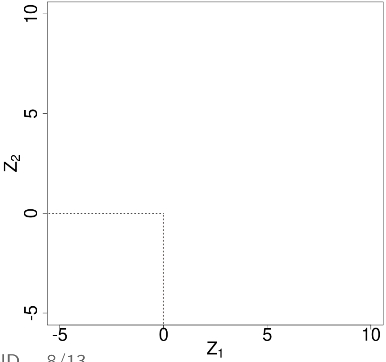

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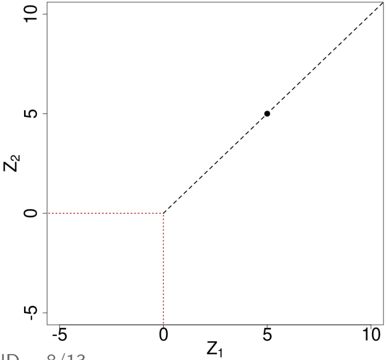

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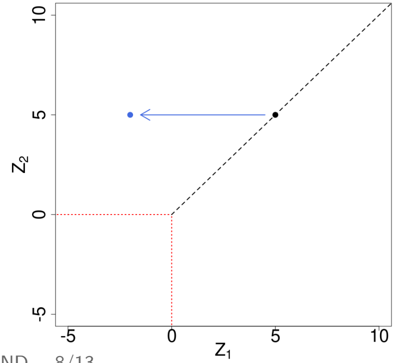

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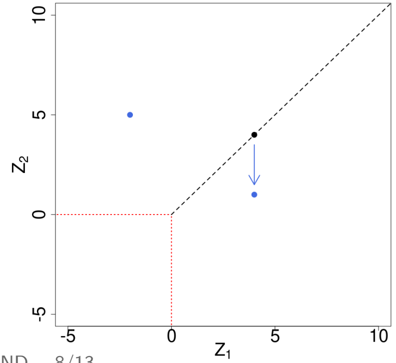

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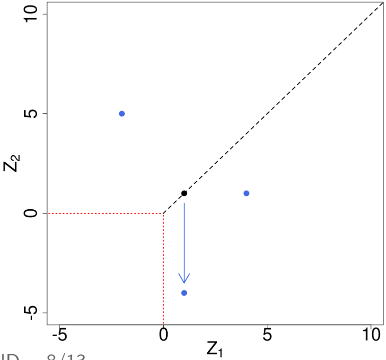

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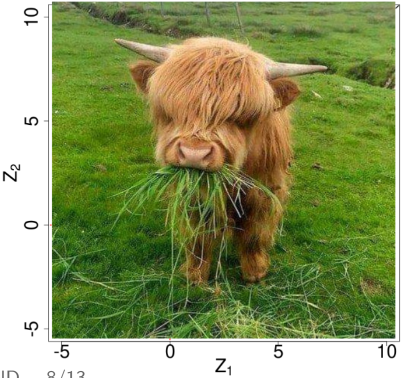

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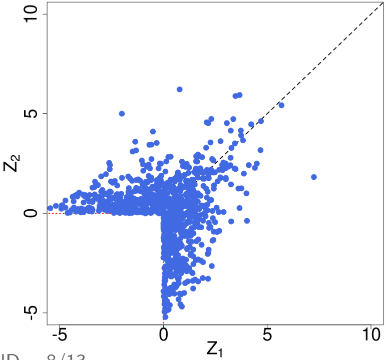

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- **Non-parametric bootstrap** to generate values of $\Delta$

## Nonparametric bootstrap MGP simulation

$$\begin{cases} Z_1 = E + \Delta \mathbb{1}_{\Delta<0}, \\ Z_2 = E - \Delta \mathbb{1}_{\Delta \geq 0} \end{cases} \tag{4}$$

- Easy to simulate $E \sim Exp(1)$
- **Non-parametric bootstrap** to generate values of $\Delta$

↙ ↘

**Joint simulation of MGP vectors**

$(Z_1, Z_2) \sim \text{MGP}$

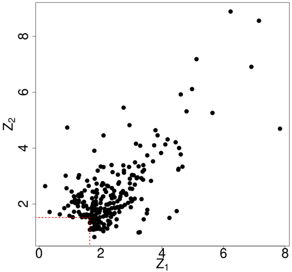
**Conditional simulation within the MGP class**

$Z_2 \mid Z_1 = z_1$ *

*(using $Z_2 = Z_1 - \Delta$)

## Application to significant wave height

**Recall** $$(Z_1, Z_2) := \left[\left(H_o^E - u_o, H_c^E - u_c\right) \mid H_o^E > u_o \text{ or } H_c^E > u_c\right] \tag{3}$$



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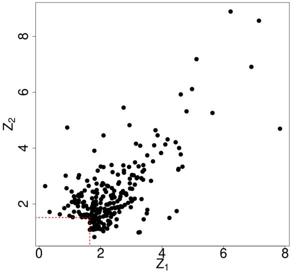


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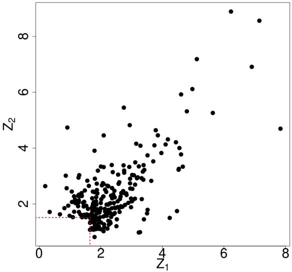


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- Then transform to the **original scale**

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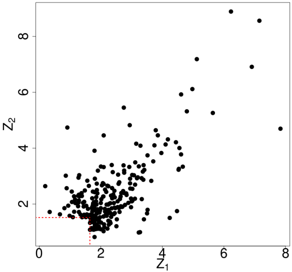


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↙ ↘

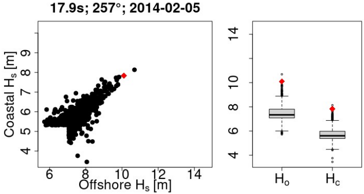
**Joint simulation**

$H_o, H_c \mid H_o > v_o, T_p, D_p$

**Conditional simulation**

$H_c \mid H_o > v_o, T_p, D_p$

## Joint simulation of extreme $H_s$




Scatterplot of simulated pairs $(H_o, H_c)$ conditionally on $(T_p, D_p)$ offshore

- Simulation sample size equal to $m = 1000$ for each observation pair
- Red dots = observed $(H_o, H_c)$ values
- Marginal distributions for each simulation setup

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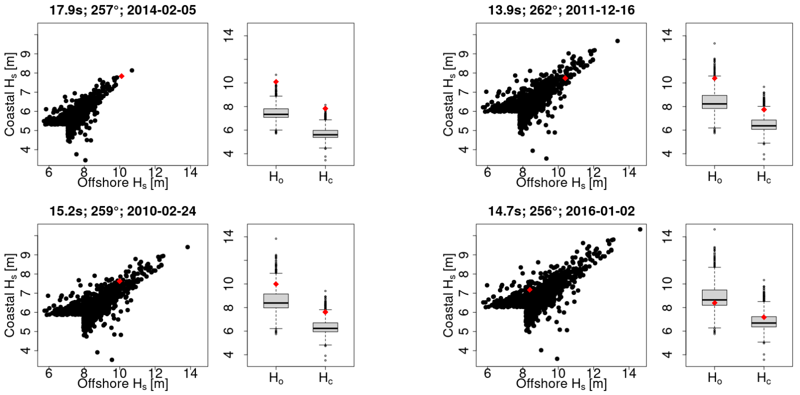


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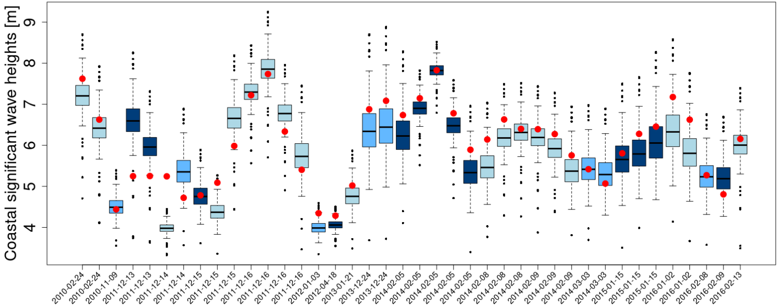
## Conditional simulation of extreme $H_s$




Boxplot of predicted $H_c$, conditionally on $(H_o, T_p, D_p)$

- Simulation sample size equal to $m = 1000$ for each observation
- Red dots = observed $H_c$ values
- Alternating colours = different storm events

## To sum up

### Contributions

- Non-stationary marginal modelling within the EGPD class
- Non-parametric generator of bivariate GP vectors
- Applied to joint and conditional simulations of extreme $H_s$

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- Dimension greater than 2
  - ⇝ from 1 coastal predictand to a field
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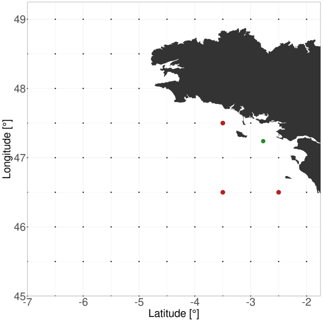

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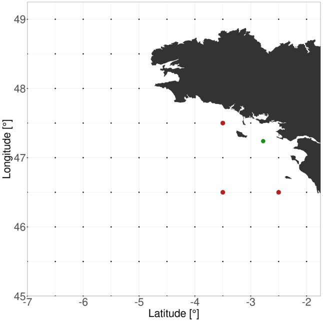

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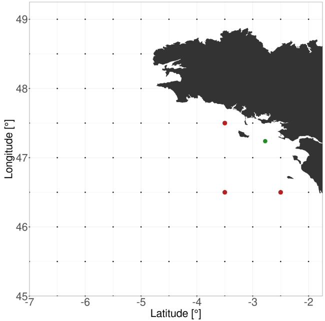


▶ More in Legrand et al., *Joint stochastic simulation of extreme coastal and offshore significant wave heights* (2023) Annals of Applied Statistics

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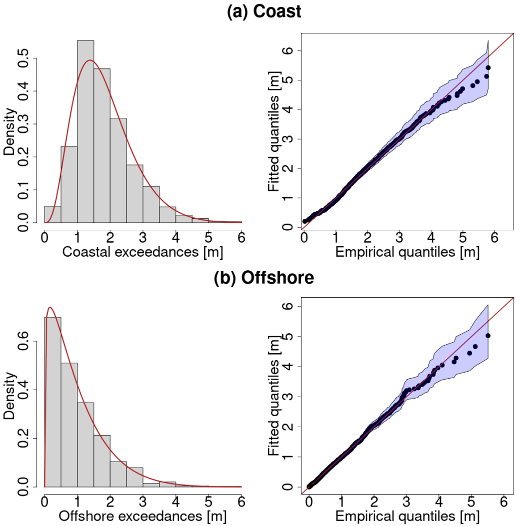
# Marginal modelling



# Marginal modelling

| Parameter | Estimate | Asymptotic CI | Bootstrap CI |
|---|---|---|---|
| $\xi_C$ | $-0.11$ | $[-0.15, -0.07]$ | $[-0.21, -0.09]$ |
| $\kappa_C$ | $4.11$ | $[3.57, 4.64]$ | $[3.02, 4.84]$ |
| $\xi_O$ | $-0.10$ | $[-0.16, -0.04]$ | $[-0.17, -0.04]$ |
| $\kappa_O$ | $1.16$ | $[1.05, 1.26]$ | $[1.05, 1.28]$ |

# Joint simulation algorithm

**Algorithm 1** Non-parametric bootstrap MGP simulation

**input** A sample $(Z_{1,i}, Z_{2,i})_{1\leq i\leq n}$ from a MGP distribution
**output** A simulated sample $(Z_{1,k}^{(m)}, Z_{2,k}^{(m)})_{1\leq k\leq m}$, potentially with $m \neq n$
**procedure**
    Define $\Delta_i := Z_{1,i} - Z_{2,i}$ for $1 \leq i \leq n$
    Generate $m$ realisations $E_k^{(m)} \sim Exp(1)$, independently of $(\Delta_i)_{1\leq i\leq n}$, for $1 \leq k \leq m$
    Bootstrap $m$ realisations $\Delta_k^{(m)}$, $1 \leq k \leq m$, from $(\Delta_1, \ldots, \Delta_n)$
**end procedure**
**return** $Z_{1,k}^{(m)} := E_k^{(m)} + \Delta_k^{(m)} \mathbb{1}_{\Delta_k^{(m)}<0}$ and $Z_{2,k}^{(m)} := E_k^{(m)} - \Delta_k^{(m)} \mathbb{1}_{\Delta_k^{(m)}>0}$, for $1 \leq k \leq m$

# Conditional simulation algorithm

**Algorithm 2** Non-parametric conditional MGP simulation

```
1: input A sample $(\Delta_i)_{1\leq i\leq n}$ ; a realisation $z_1$ of $Z_1$
2: output A simulated sample $(Z_{2,k}^{(m)})_{1\leq k\leq m}$ conditionally on $Z_1 = z_1$, potentially with $m \neq n$
3: procedure
4:     if $z_1 > 0$ then
5:         Define $\Delta_{|Z_1^+}$ the subset of $(\Delta_i)_{1\leq i\leq n}$ such that $Z_1 > 0$
6:         Bootstrap $m$ realisations $\Delta_k^{(m)}$, $1 \leq k \leq m$, from $\Delta_{|Z_1^+}$ independently of $Z_1$
7:     else
8:         for $1 \leq k \leq m$ do
9:             Sample one realisation $\Delta_k^{(m)}$ from $(\Delta_i)_{1\leq i\leq n}$ independently of $Z_1$
10:            Generate a random number $u \in [0, 1]$
11:            while $u > \exp(\Delta_k^{(m)}) \mathbb{1}_{\Delta_k^{(m)} < z_1}$ do
12:                Repeat steps 9 and 10
13:        end for
14: end procedure
15: return $Z_{2,k}^{(m)} := z_1 - \Delta_k^{(m)}$ for $1 \leq k \leq m$
```

## Numerical experiments: $(T_1, T_2) \sim \mathcal{N}((\mu_1, \mu_2), \Sigma)$, $\mu_1 \neq \mu_2$ (1)

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = E + \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} - \max(T_1, T_2)$$

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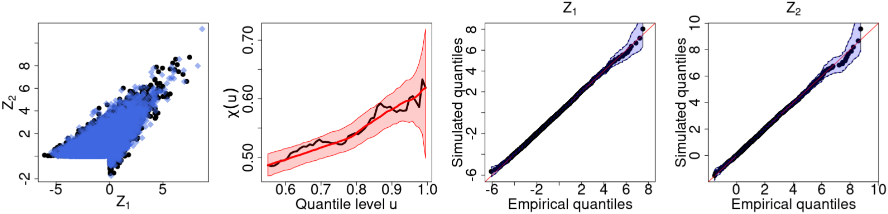
Joint simulation:



Numerical experiments: $(T_1, T_2) \sim \mathcal{N}((\mu_1, \mu_2), \Sigma)$, $\mu_1 \neq \mu_2$ (2)

Conditional simulation $Z_2 \mid Z_1 = z_1$:

## Numerical experiments: $(T_1, T_2) \sim \mathcal{N}((\mu_1, \mu_2), \Sigma)$, $\mu_1 \neq \mu_2$ (2)

Conditional simulation $Z_2 \mid Z_1 = z_1$: (theoretical conditional density in red)