Markov switching Hawkes process

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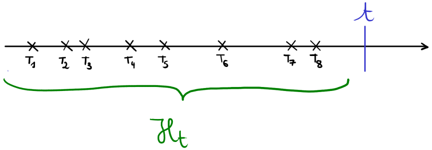
Point process

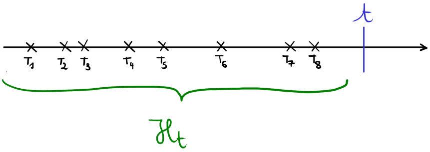


- $(T_k)_{k \geqslant 1}$ a random collection of points
- Intensity function $\lambda(t)$: immediate probability of observing an event at time t

Hawkes process

The intensity function depends on the past history $\mathcal{H}_t$.

Linear univariate Hawkes process (Hawkes, 1971)



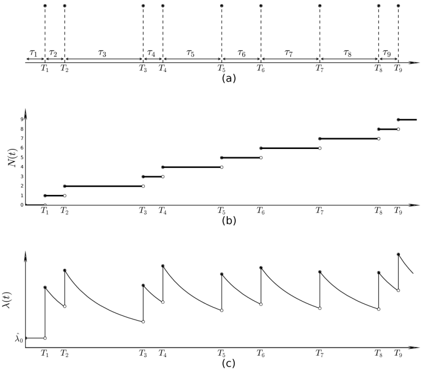
(Conditional) intensity function

$$\lambda(t|\mathcal{H}_t) = \lambda(t) = \lambda_0 + \sum_{T_k \leqslant t} h(t - T_k)$$

- λ_0 is called the baseline
- h describes the influence of past events

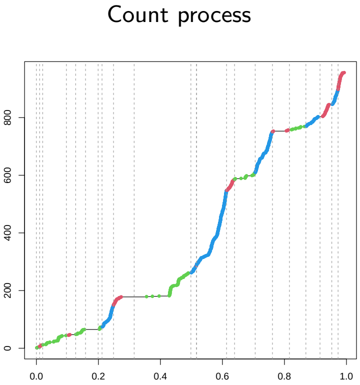
Self-exciting exponential Hawkes process

$$\lambda(t) = \lambda_0 + \sum_{T_k \leqslant t} a e^{-b(t-T_k)}$$



- Classical exponential kernel function $h(t) = ae^{-bt}$
- $a \geqslant 0$, which ensures that λ is non negative
- An event increases the probability of observing another event
- Historical applications: epidemiology, sismology

Modeling different phases



Goals

- New modeling to integrate different phases while keeping the Hawkes dependence structure
- Estimating all parameters and the change points

Applications: when covariates are observed

- Influence of external covariates on fishing behaviour of narwhals (collaboration with C. Dion, M. Sadeler, A. Samson)

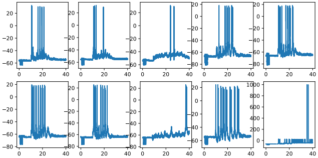
$$\lambda(t) = \lambda_0(X_t) + \sum_{T_k \leqslant t} h(t - T_k, X_t)$$

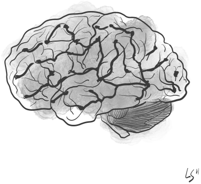
Observations

- $(T_k)_{k \geqslant 1}$ times of sounds emissions while fishing
- Covariates: X_t sound exposure, sheep position at time t

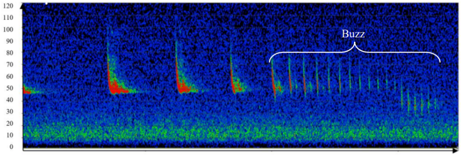
Applications

- Neuroscience:
  - Study neuronal activity through spike trains
  - Different phases of brain activity





- Ecology:
  - Understand animal behaviours with sound recordings
  - Movement/hunting phases



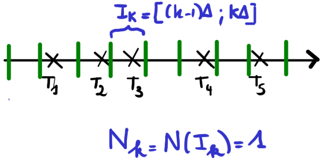
Denis et al. (2023)

Discrete time Hawkes process

Exponential Hawkes process

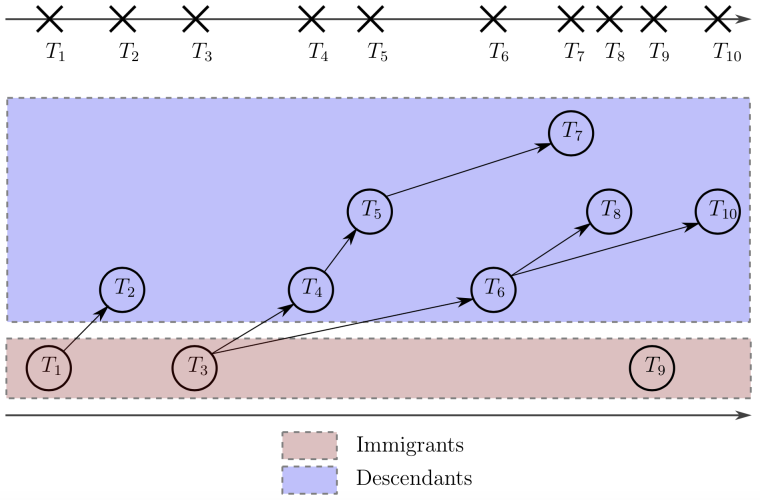
$$\lambda(t) = \lambda_0 + \sum_{T_k < t} a e^{-b(t - T_k)}$$

- $I_k = [\tau_{k-1}; \tau_k]$ with $\tau_k = k\Delta$
- $N_k = N(I_k)$ the number of events on I_k



- Distribution of N_k?

Cluster representation (Hawkes and Oakes, 1974)



- Immigrants arrive at rate λ_0
- All individuals T (immigrant and descendant) produce a new individual at rate $h(t - T)$

Discrete time Hawkes process

Branching structure

$$N_k \triangleq B_k + \sum_{\ell \leqslant k-1} \sum_{T \in I_\ell} M_T(I_k) + R_k$$

- $B_k \sim \mathcal{P}(\lambda_0 \Delta)$ discrete immigrant process
- $M_T(I_k) \sim \mathcal{P}\left(c(a, b, \Delta) e^{-b(\tau_{k-1} - T)}\right)$ descendants of $T < \tau_{k-1}$
- R_k number of descendants of points $T \in I_k$ (neglected if Δ small)

Approximation of N_k

$$Y_k \mid \{Y_\ell\}_{\ell \leqslant k-1} \sim \mathcal{P}\left(\mu + \sum_{\ell=1}^{\infty} \alpha \beta^\ell Y_{k-\ell}\right)$$

with $\mu = \lambda_0 \Delta$ and α, β depending on a, b, Δ.

Markovian reformulation

Approximation of N_k

$$Y_k \mid \{Y_\ell\}_{\ell \leqslant k-1} \sim \mathcal{P}\left(\mu + \sum_{\ell=1}^{\infty} \alpha \beta^\ell Y_{k-\ell}\right)$$

- $\{Y_k\}_{k \geqslant 1}$ is not a Markov chain
- If we define U_k such that

$$\begin{cases} U_1 = 0 \\ U_k = \alpha Y_{k-1} + \beta U_{k-1} \end{cases}$$

$\rightarrow (Y_k, U_k)_{\geqslant 1}$ is a Markov Chain

Discrete time Hawkes HMM

- Hidden path: $\{Z_k\}_{k \geqslant 1}$ homogeneous Markov chain with transition matrix π
- Observed counts: for $k \geqslant 1$, set $U_1 = 0$ and

$$Y_k \mid \{Y_\ell\}_{\ell \leqslant k-1} \sim \mathcal{P}\left(\mu_{Z_k} + \sum_{\ell=1}^{\infty} \alpha \beta^\ell Y_{k-\ell}\right)$$

  - The immigration rate varies with the hidden path
  - The number of offspring does not vary with the hidden path

$$\begin{tikzcd}
{} \arrow[rr, dashed] & & Z_{k-1} \arrow[rr] \arrow[dd] & & Z_k \arrow[rr] \arrow[dd] & & Z_{k+1} \arrow[rr, dashed] \arrow[dd] & & {} \\
& U_{k-1} \arrow[rr] \arrow[dr] & & U_k \arrow[rr] \arrow[dr] & & U_{k+1} \arrow[rr] \arrow[dr] & & U_{k+2} \arrow[r, dashed] & {} \\
{} \arrow[ur, dashed] & & Y_{k-1} \arrow[ur] & & Y_k \arrow[ur] & & Y_{k+1} \arrow[ur] & &
\end{tikzcd}$$

Inference

Goals

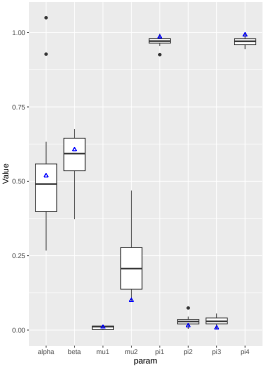
- Estimate $\mu = (\mu_1, \ldots, \mu_Q)$, α, β, π
- Learn $p(Z|Y) \rightarrow$ detect the change points

- Introducing $(U_k)_{k \geqslant 1} \rightarrow$ regular EM algorithm for HMM
- M-step: Recurrence formulas for the gradients
- 3-step initialization:
  - Homogeneous Hawkes for the reproduction parameters α and β
  - Poisson-HMM for the rates $\mu_1, \ldots, \mu_Q$
  - Correction: $\widetilde{\mu}_k = \mu_k(1 - a/b)$ to account for reproduction rate

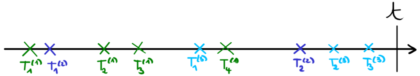
Numerical results

Classification performance

	States	1	2
Hawkes HMM	1	285.0	43.85
	2	160.6	510.55
Poisson HMM	1	312.00	16.85
	2	491.95	179.20



Extension I : Multivariate Hawkes process



Conditional intensity

$$\lambda^{(i)}(t) = \lambda_0^i + \sum_{i=1}^{q} \sum_{T_k^j \leqslant t} h_{i,j}(t - T_k^j)$$

- One subprocess for each neuron/species
- Allows to model interactions

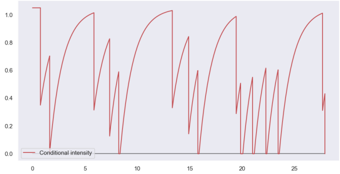
Extension II : Changing memory/interaction functions

- The hidden path can influence the memory/interaction functions
- U_k depends on Z_{k-1}

$$
\begin{tikzcd}
{} \arrow[rr, dashed] \arrow[dr, dashed] & & Z_{k-1} \arrow[rr] \arrow[dd] \arrow[dr] & & Z_k \arrow[rr] \arrow[dd] \arrow[dr] & & Z_{k+1} \arrow[rr, dashed] \arrow[dd] \arrow[dr] & & {} \\
& U_{k-1} \arrow[rr] \arrow[dr] & & U_k \arrow[rr] \arrow[dr] & & U_{k+1} \arrow[rr] \arrow[dr] & & U_{k+2} \arrow[r, dashed] & {} \\
{} \arrow[ur, dash, dashed] & & Y_{k-1} \arrow[ur] & & Y_k \arrow[ur] & & Y_{k+1} \arrow[ur] & &
\end{tikzcd}
$$

Extension III : Nonlinear Hawkes process

- Modeling inhibition ($h \leqslant 0$)



$$\lambda(t) = \phi\left(\lambda_0 + \sum_{T_k \leqslant t} h(t - T_k)\right)$$

- Loss of the branching structure